2019

AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356,$ and $E(t) - L(t) < 0$ for $6.20356 < t \le 8.$ Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$
$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$3 : \begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0,$ the rate of change in the number of fish is decreasing at time $t = 5.$

$2 : \begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

$\int_0^5 E(t)\,dt \approx 153$ fish

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

$\frac{1}{5}\int_0^5 L(t)\,dt \approx \frac{30.295}{5} \approx 6.059$ fish per hour leave the lake

Continue question 1 on page 5.

(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

$E(t)-L(t)=0$

$t \approx 6.204$

$E(t)-L(t)$ is the rate at which the number of fish is changing

At time $t=6.204$, the greatest number of fish in the 8 hour period are in the lake. This is because $E(t)-L(t)$ is positive from $t=0$ to $t=6.204$ indicating that the number of fish in the lake is increasing over $(0, 6.204)$, but $E(t)-L(t)$ is negative from $t=6.204$ to $t=8$, which means the number of fish are decreasing in this time period, so the number of fish in the lake is greatest at $t=6.204$ hours

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t=5$)? Explain your reasoning.

$\frac{d}{dt}\left(16+15\sin\left(\frac{\pi t}{6}\right)-2^{0.1t^2}\right)\Big|_{t=5} \approx -10.723$ fish/hour^2

Since the derivative of $E(t)-L(t)$ at $t=5$ is negative, the rate of change in the number of fish in the lake is decreasing

1B
1 of 2

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

$$\int_0^5 E(t)\,dt = 153.458$$

153 fish enter over 5-hour period

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

$$\frac{1}{5}\int_0^5 L(t)\,dt = 6.059$$

Continue question 1 on page 5.

1B
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(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

max

$E(t) - L(t) = 0$ and changes signs (+) to (−)

$t = 6.204$

There is the greatest number of fish in the lake at time $t = 6.204$ hours

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$) ? Explain your reasoning.

$E(5) - L(5) = 17.843$

The rate of change in the number of fish in the lake is increasing since $E(5) - L(5) > 0$ and represents the rate of change.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

(a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ? Give your answer to the nearest whole number.

$\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt$ ~~$\int_0^5 4 + 2^{0.1t^2}$~~

153.458 or $\boxed{153}$

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$) ?

$\int_0^5 \left(4 + 2^{0.1t^2}\right)dt \rightarrow \boxed{30.295}$

Continue question 1 on page 5.

1C
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(c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

~~8~~ 5 because $\int_0^8 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \int_0^8 \left(4 + 2^{0.1t^2}\right)dt$ 80.92

and $\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \int_0^8 \left(4 + 2^{0.1t^2}\right)dt = 123.163$

$123.163 > 80.92$

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

Increasing because at 4 A.M. 100

$\left(\int_0^4 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt \int_0^4 \left(4 + 2^{0.1t^2}\right)dt = 100.837 \right)$

fish are in the lake but at

5 A.M. → 123.163 fish are in the

lake $\left(\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt - \int_0^5 \left(4 + 2^{0.1t^2}\right)dt = 123.163 \right)$

$123.163 > 100.837$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem, fish enter and leave a lake at rates modeled by functions E and L given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$ and $L(t) = 4 + 2^{0.1t^2}$, respectively. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight $(t = 0)$.

In part (a) students were asked to find the number of fish entering the lake between midnight $(t = 0)$ and 5 A.M. $(t = 5)$ and to provide the answer rounded to the nearest whole number. A response should demonstrate an understanding that a definite integral of the rate at which fish enter the lake over the time interval $0 \le t \le 5$ gives the number of fish that enter the lake during that time period. The numerical value of the integral $\int_0^5 E(t)\,dt$ should be obtained using a graphing calculator.

In part (b) students were asked for the average number of fish that leave the lake per hour over the 5-hour period $0 \le t \le 5.$ A response should demonstrate that "number of fish per hour" is a rate, so the question is asking for the average value of $L(t)$ across the interval $0 \le t \le 5,$ found by dividing the definite integral of L across the interval by the width of the interval. The numerical value of the expression $\frac{1}{5}\int_0^5 L(t)\,dt$ should be obtained using a graphing calculator.

In part (c) students were asked to find, with justification, the time t in the interval $0 \le t \le 8$ when the population of fish in the lake is greatest. The key understanding here is that the rate of change of the number of fish in the lake, in number of fish per hour, is given by the difference $E(t) - L(t).$ Analysis of this difference using a graphing calculator shows that, for $0 \le t \le 8,$ the difference has exactly one sign change, occurring at $t = 6.20356.$ Before this time, $E(t) - L(t) > 0,$ so the number of fish in the lake is increasing; after this time, $E(t) - L(t) < 0,$ so the number of fish in the lake is decreasing. Thus the number of fish in the lake is greatest at $t = 6.204$ (or 6.203). An alternative justification uses the definite integral of $E(t) - L(t)$ over an interval starting at $t = 0$ to find the net change in the number of fish in the lake from time $t = 0.$ The candidates for when the fish population is greatest are the endpoints of the time interval $0 \le t \le 8$ and the one time when $E(t) - L(t) = 0,$ namely $t = 6.20356.$ Numerical evaluation of the appropriate definite integrals on a graphing calculator shows that the number of fish in the lake is greatest at $t = 6.204$ (or 6.203).

In part (d) students were asked whether the rate of change in the number of fish in the lake is increasing or decreasing at time $t = 5.$ A response should again demonstrate the understanding that the rate of change of the number of fish in the lake is given by the difference $E(t) - L(t),$ and whether this rate is increasing or decreasing at time $t = 5$ can be determined by the sign of the derivative of the difference at that time. Using a graphing calculator to find that $E'(5) - L'(5) < 0$ leads to the conclusion that the rate of change in the number of fish in the lake is decreasing at time $t = 5.$

Question 1 (continued)

For part (a) see LO CHA-4.E/EK CHA-4.E.1, LO LIM-5.A/EK LIM-5.A.3. For part (b) see LO CHA-4.B/EK CHA-4.B.1. For part (c) see LO FUN-4.B/EK FUN-4.B.1. For part (d) see LO CHA-3.C/EK CHA-3.C.1, LO CHA-2.D/EK CHA-2.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 1A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 E(t)\,dt,$ and the second point was earned with the answer $153.$ In part (b) the first point was earned with the definite integral $\int_0^5 L(t)\,dt.$ The second point was earned with multiplying the integral by $\frac{1}{5}$ and with the answer 6.059 that is accurate to three decimal places. In part (c) the first point was earned with the equation $E(t) - L(t) = 0$ in line 1 on the left. The sentence "[a]t time $t = 6.204,$ the greatest number of fish in the 8 hour period are in the lake" in lines 3 and 4 would have earned the second point without additional information. The second point was earned with the restatement "so the number of fish in the lake is greatest at $t = 6.204$ hours" in lines 7 and 8. The third point was earned with the statements "because $E(t) - L(t)$ is positive from $t = 0$ to $t = 6.204$" and "$E(t) - L(t)$ is negative from $t = 6.204$ to $t = 8$" in lines 4, 5, and 6. In part (d) the response earned the first point in line 1 with $16 + 15\sin\left(\frac{\pi t}{6}\right) - 2^{0.1t^2},$ which is equivalent to $E(t) - L(t),$ and $\frac{d}{dt}\left(16 + 15\sin\left(\frac{\pi t}{6}\right) - 2^{0.1t^2}\right)\Big|_{t=5},$ which is equivalent to $E'(5) - L'(5).$ The second point was earned with "decreasing" and the explanation, "[s]ince the derivative of $E(t) - L(t)$ at $t = 5$ is negative" in the concluding sentence.

Sample: 1B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 E(t)\,dt,$ and the second point was earned with the answer $153.$ In part (b) the first point was earned with the definite integral $\int_0^5 L(t)\,dt.$ The second point was earned with multiplying the integral by $\frac{1}{5}$ and with the answer 6.059 that is accurate to three decimal places. In part (c) the first point was earned with the equation $E(t) - L(t) = 0$ in line 1. The second point was earned with "[t]here is the greatest number of fish in the lake at time $t = 6.204$" in lines 3 and 4. The third point was not earned because the statement "changes signs $(+)$ to $(-)$" and $t = 6.204$ in lines 1 and 2 only provides evidence that there is a relative maximum at $t = 6.204$ rather than an absolute maximum on the interval $0 \le t \le 8.$ In part (d) no points were earned because there is no mention of $E'(5)$ and $L'(5),$ and the answer of "increasing" is incorrect.

Question 1 (continued)

Sample: 1C
Score: 3

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the first point was earned with the definite integral $\int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right) dt,$ and the second point was earned with the answer $153.$ The crossed-out work is not scored. In part (b) the first point was earned with the definite integral $\int_0^5 \left(4 + 2^{0.1t^2}\right) dt.$ The second point was not earned because the integral is not multiplied by $\frac{1}{5};$ the answer is incorrect. In part (c) no points were earned because there is no equating of $E(t) - L(t)$ to $0,$ and there is no declaration of an absolute maximum value nor a justification. In part (d) no points were earned because there is no mention of $E'(5)$ and $L'(5),$ and the answer of "Increasing" is incorrect.

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Sample Student Responses and Scoring Commentary

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Question 2

(a) v_P is differentiable $\Rightarrow$ v_P is continuous on $0.3 \le t \le 2.8$.

$$\frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0$$

By the Mean Value Theorem, there is a value c, $0.3 < c < 2.8$, such that $v_P'(c) = 0$.

— OR —

v_P is differentiable $\Rightarrow$ v_P is continuous on $0.3 \le t \le 2.8$.

By the Extreme Value Theorem, v_p has a minimum on $[0.3, 2.8]$.

$v_P(0.3) = 55 > -29 = v_P(1.7)$ and $v_P(1.7) = -29 < 55 = v_P(2.8)$.

Thus v_P has a minimum on the interval $(0.3, 2.8)$.

Because v_P is differentiable, $v_P'(t)$ must equal 0 at this minimum.

$$2 : \begin{cases} 1 : v_P(2.8) - v_P(0.3) = 0 \\ 1 : \text{justification, using} \\ \quad \text{Mean Value Theorem} \end{cases}$$

— OR —

$$2 : \begin{cases} 1 : v_P(0.3) > v_P(1.7) \\ \quad \text{and } v_P(1.7) < v_P(2.8) \\ 1 : \text{justification, using} \\ \quad \text{Extreme Value Theorem} \end{cases}$$

(b)
$$\begin{aligned} \int_0^{2.8} v_P(t)\,dt &\approx 0.3\left(\frac{v_P(0) + v_P(0.3)}{2}\right) + 1.4\left(\frac{v_P(0.3) + v_P(1.7)}{2}\right) \\ &\quad + 1.1\left(\frac{v_P(1.7) + v_P(2.8)}{2}\right) \\ &= 0.3\left(\frac{0 + 55}{2}\right) + 1.4\left(\frac{55 + (-29)}{2}\right) + 1.1\left(\frac{-29 + 55}{2}\right) \\ &= 40.75 \end{aligned}$$

1 : answer, using trapezoidal sum

(c) $v_Q(t) = 60 \Rightarrow t = A = 1.866181$ or $t = B = 3.519174$

$v_Q(t) \ge 60$ for $A \le t \le B$

$$\int_A^B \left|v_Q(t)\right|\,dt = 106.108754$$

The distance traveled by particle Q during the interval $A \le t \le B$ is 106.109 (or 106.108) meters.

$$3 : \begin{cases} 1 : \text{interval} \\ 1 : \text{definite integral} \\ 1 : \text{distance} \end{cases}$$

(d) From part (b), the position of particle P at time $t = 2.8$ is

$$x_P(2.8) = \int_0^{2.8} v_P(t)\,dt \approx 40.75.$$

$$x_Q(2.8) = x_Q(0) + \int_0^{2.8} v_Q(t)\,dt = -90 + 135.937653 = 45.937653$$

Therefore at time $t = 2.8$, particles P and Q are approximately $45.937653 - 40.75 = 5.188$ (or 5.187) meters apart.

$$3 : \begin{cases} 1 : \int_0^{2.8} v_Q(t)\,dt \\ 1 : \text{position of particle } Q \\ 1 : \text{answer} \end{cases}$$

2A 1of2

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	−29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

$V_P'(t) = a_P(t) \qquad \frac{V_P(2.8) - V_P(0.3)}{2.8 - 0.3} = V_P'(t)$

$\boxed{\text{Since } V_P(t) \text{ is a continuous and differentiable function, MVT states that there must be at least one time } t, \text{ for } 0.3 \le t \le 2.8 \text{ at which } V_P'(t) = \frac{V_P(2.8) - V_P(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0.}$

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

$0.3\left(\frac{55+0}{2}\right) + 1.4\left(\frac{-29+55}{2}\right) + 1.1\left(\frac{55+(-29)}{2}\right)$

$8.25 + 18.2 + 14.3$

$\boxed{\int_0^{2.8} V_P(t)\,dt = 40.75}$

Continue question 2 on page 7.

2A 2 of 2

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

$$60 = 45\sqrt{t}\cos(0.063t^2)$$

$$t = 1.866,\ 3.519$$

$$\boxed{1.866 \le t \le 3.519}$$

$$\int_{1.866}^{3.519} 45\sqrt{t}\cos(0.063t^2)\,dt = 106.109$$

$$\boxed{106.109 \text{ meters}}$$

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

$$x_P(2.8) = 40.75$$

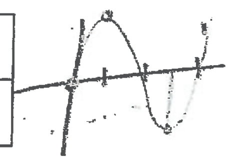
$$x_Q(2.8) = -90 + \int_0^{2.8} 45\sqrt{t}\cos(0.063t^2)\,dt$$

$$= -90 + 135.938 = 45.938$$

$$45.938 - 40.75 = 5.188$$

The distance between particles P and Q at t = 2.8 is 5.188 meters.

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	−29	55	48



2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

$V_p(t)$ is differentiable and continuous

and $V_p(.3)=55$ and $V_p(2.8)=55$ $\therefore$

by Rolle's Theorem there exists c value where

$V_p'(c)=0$

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

$$\int_0^{2.8} v_p(t)\,dt \approx .3\left(\frac{0+55}{2}\right) + 1.4\left(\frac{84}{2}\right) + 1.1\left(\frac{84}{2}\right)$$

$\approx$ 113.25 meters

Continue question 2 on page 7.

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

$V_Q(t) \geq 60$

$t = 1.86618$

the velocity of particle Q is at least 60 meters per hour on $(1.86618, 4]$ hours.

$\int_{1.86618}^{4} |V_Q(t)|\, dt \approx 132.369$ meters

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

$\int v(t) = s(t)$

$-90 + \int_0^{2.8} V_Q(t) \approx 45.9377$ meters

$\int_0^{2.8} v_P(t)\,dt \approx 113.25$ meters

Particles P and Q are ≈ 67.312 meters apart from each other

2C 1of2

t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	−29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

$V_p(t)$ represents the velocity (first derivative) of the particle. From $0.3 \le t \le 2.8$, we see the velocity change signs twice, which represents the graph of $V_p(t)$ moving across the x-axis twice, Therefore by crossing the x-axis (twice), there must be a time where $V'p(t)$ equals 0.

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

$$\frac{2.8-0}{2(3)}\left[55 + 2(26) + 26\right] = \boxed{62.067}$$

Continue question 2 on page 7.

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

$v_Q(0) = 0$
$V_Q(1) = 44.911$
$V_Q(2) = 61.630$
$V_Q(3) = 65.746$
$V_Q(4) = 48.020$

Velocity of Q is at least 60 m/hour at $(2 \le t \le 3)$

$\int_2^3 |V_Q(t)|\,dt =$ 65.036 m traveled when velocity of Q is at least 60 m/hour

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

At $t = 2.8 \rightarrow P(t)$ is 62.067

$\int_0^{2.8} V_Q(t)\,dt = 135.938 =$ Q position

$\int_0^{2.8} V_P(t)\,dt = 62.067$

$135.938 - 62.067 =$ 73.871 m between P and Q at $t = 2.8$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a particle P moves along the x-axis with velocity given by a differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. The particle starts at the origin at time $t = 0$, and selected values of $v_P(t)$ are given in a table.

In part (a) students were asked to justify why there is at least one time t, for $0.3 \le t \le 2.8$, when the acceleration of particle P is 0. A response should demonstrate that the hypotheses of the Mean Value Theorem are satisfied on the given interval and that applying the Mean Value Theorem to v_P on $[0.3, 2.8]$ leads to the desired conclusion.

In part (b) students were asked to approximate $\int_0^{2.8} v_P(t)\,dt$ using a trapezoidal sum and data from the table of selected values of $v_P(t)$. A response should demonstrate the form of a trapezoidal sum using the three subintervals indicated.

In part (c) a second particle, Q, is introduced, also moving along the x-axis, and with velocity $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Students were asked to find the time interval during which $v_Q(t) \ge 60$ and to find the distance traveled by particle Q during this time interval. Using a graphing calculator to find the interval, a response should demonstrate that the distance traveled by particle Q is given by the definite integral of the absolute value of v_Q over this time interval. The value of this integral is found using the numerical integration capability of a graphing calculator.

In part (d) students were given that particle Q starts at position $x = -90$ at time $t = 0$ and were asked to use the approximation from part (b) and the velocity function v_Q introduced in part (c) to approximate the distance between particles P and Q at time $t = 2.8$. A response should demonstrate that the integral approximated in part (b) gives the position of particle P at time $t = 2.8$, and that the position of particle Q at this time is found by adding the particle's initial position, $x = -90$, to $\int_0^{2.8} v_Q(t)\,dt$. The student's response should report the difference between these two positions.

For part (a) see LO CHA-2.A/EK CHA-2.A.1, LO FUN-1.B/EK FUN-1.B.1. For part (b) see LO LIM-5.A/EK LIM-5.A.2. For part (c) see LO CHA-4.C/EK CHA-4.C.1, LO LIM-5.A/ EK LIM-5.A.3. For part (d) see LO CHA-4.C/EK CHA-4.C.1, LO LIM-5.A/EK LIM-5.A.3. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 2A
Score: 9

The response earned 9 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the response would have earned the first point in line 3 of the boxed work by presenting the difference

Question 2 (continued)

quotient $\frac{55-55}{2.5}$. Numerical simplification is not required. The response simplifies the difference quotient to 0 in line 3 of the boxed work, and the first point was earned. In lines 1, 2, and 3 of the boxed work, the response earned the second point with "[s]ince $v_P(t)$ is a continuous and differentiable function, MVT states that there must be at least one time t, for $0.3 \le t \le 2.8$ at which $v_P{}'(t) = \frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = \frac{55-55}{2.5} = 0$." "MVT" is an acceptable form of communication for the Mean Value Theorem. In part (b) the response would have earned the point in line 1 with the trapezoidal sum $0.3\left(\frac{55+0}{2}\right) + 1.4\left(\frac{-29+55}{2}\right) + 1.1\left(\frac{55+(-29)}{2}\right)$. Numerical simplification is not required. The response simplifies the expression to 40.75 in the boxed work and earned the point. In part (c) the response earned the first point in line 3 with the boxed interval $1.866 \le t \le 3.519$. The second point was earned in line 1 on the left with the definite integral $\int_{1.866}^{3.519} 45\sqrt{t}\cos\left(0.063t^2\right)\,dt$. The response earned the third point with the boxed distance 106.109 meters on the right. Units are not required to earn the point. In part (d) the response earned the first point in line 2 with the definite integral $\int_0^{2.8} 45\sqrt{t}\cos\left(0.063t^2\right)\,dt$. The response would have earned the second point in line 3 with the position of particle Q at $t = 2.8$ is $-90 + 135.938$. Numerical simplification is not required. The response simplifies to 45.938 and earned the second point. The response would have earned the third point in line 4 for the difference $45.938 - 40.75$. Numerical simplification is not required. The response simplifies to 5.188, restates this value in the box, and earned the third point. Units are not required to earn the point.

Sample: 2B
Score: 6

The response earned 6 points: 2 points in part (a), no point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the response earned the first point in line 2 with the equations $v_P(.3) = 55$ and $v_P(2.8) = 55$. The response earned the second point in lines 1, 3, and 4 with "$v_P(t)$ is differentiable and continuous" and "by Rolle's Theorem there exists c value where $v_P{}'(c) = 0$." In part (b) the response did not earn the point because of arithmetic errors in the second and third terms of the trapezoidal sum. This results in an incorrect approximation. In part (c) the response did not earn the first point because of an incorrect interval $(1.86618, 4]$ in line 3 on the right. The response earned the second point in line 4 with the definite integral $\int_{1.86618}^{4} \left|v_Q(t)\right|\,dt$ based on an eligible incorrect interval of $(1.86618, 4]$. The response did not earn the third point because of an incorrect distance. In part (d) the response earned the first point in line 2 with the definite integral $\int_0^{2.8} v_Q(t)$. The missing dt does not impact earning the point. The response earned the second point in line 2 with the position of particle Q at $t = 2.8$ is 45.9377 meters. Units are not required to earn the point. The response earned the third point in line 4 with the position of particle Q at $t = 2.8$ and the imported incorrect value 113.25 from part (b), which are used to compute the consistent answer of 67.312 meters. Units are not required to earn the point.

Question 2 (continued)

Sample: 2C
Score: 3

The response earned 3 points: no points in part (a), no point in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the response did not earn the first point because there is no correct difference, difference quotient, or comparison of velocities of particle P at times $t = 0.3,$ $t = 1.7,$ or $t = 2.8.$ The response is not eligible to earn the second point because the first point was not earned. In part (b) the response did not earn the point because of an incorrect trapezoidal sum. In part (c) the response did not earn the first point because an incorrect interval $2 \le x \le 3$ is presented in line 2 on the right. The response earned the second point in line 6 for the definite integral $\int_2^3 \left| v_Q(t) \right| dt$ based on an eligible incomplete interval of $2 \le x \le 3.$ The response did not earn the third point because of an incorrect distance. In part (d) the response earned the first point in line 2 with the definite integral $\int_0^{2.8} v_Q(t)\, dt.$ The response did not earn the second point because the initial condition -90 is not used to determine the position of particle Q at $t = 2.8.$ The response would have earned the third point in line 4 with the identified position of particle Q at $t = 2.8$ and the imported incorrect value 62.067 from part (b), which are used to compute the consistent answer of $135.938 - 62.067.$ Numerical simplification is not required. The response simplifies to 73.871 m and earned the point. Units are not required to earn the point.

2019

AP® CollegeBoard

AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Free Response Question 3

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$3 : \begin{cases} 1 : \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1 : \int_{-2}^{5} f(x)\,dx \\ 1 : \text{answer} \end{cases}$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$= 2(f(5) - f(3)) + 4(5 - 3)$

$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$

$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$

$= (2f(5) + 20) - (2f(3) + 12)$

$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$

$= 2 + 2\sqrt{5}$

$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

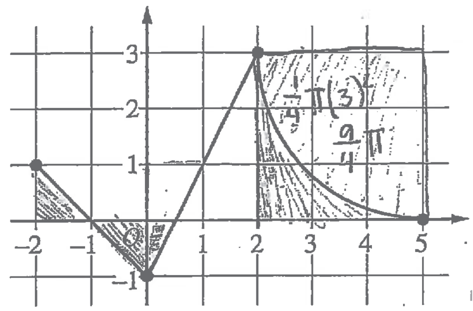
$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

3A 1 of 2



$3 \cdot \frac{3}{2} = \frac{9}{2} \cdot \frac{1}{2} = \frac{9}{4}$

Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$$

$$\int_{-6}^{-2} f(x)\,dx = 7 - \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$$\int_{3}^{5} 2f'(x) + 4\,dx = \int_{3}^{5} 2(f'(x) + 2)\,dx$$

$$= 2\int_{3}^{5} f'(x) + 2\,dx$$

$$= 2 \cdot \left[f(x) + 2x\right]_{3}^{5}$$

$$= 2 \cdot \left[\left[f(5) + 2(5)\right] - \left[f(3) + 2(3)\right]\right]$$

$$= 2 \cdot \left[(0 + 10) - \left((3 - \sqrt{5}) + 6\right)\right]$$

Continue question 3 on page 15

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$g'(x) = f(x)$

$g'(x) = f(x) = 0$

$x = -1$, $x = \frac{1}{2}$, $x = 5$

Since $f(x)$ is continuous on $-2 \le x \le 5$, then $g(x) = \int_{-2}^{x} f(t)\,dt$ is also continuous on $-2 \le x \le 5$. Therefore, by the EVT, $g(x)$ has an absolute max. on $-2 \le x \le 5$.

Candidates for abs. max: $x = -2, -1, \frac{1}{2}, 5$

$x = -2: \int_{-2}^{-2} f(t)\,dt = 0$

$x = -1: \int_{-2}^{-1} f(t)\,dt = \frac{1}{2}$

$x = \frac{1}{2}: \int_{-2}^{1/2} f(t)\,dt = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} = -\frac{1}{4}$

$x = 5: \int_{-2}^{-5} f(t)\,dt = -\frac{1}{4} + \frac{9}{4} + (9 - \frac{9\pi}{4}) = 2 + 9 - \frac{9\pi}{4} = 11 - \frac{9\pi}{4}$

The absolute maximum value of g is $11 - \frac{9\pi}{4}$ and the absolute maximum occurs at $x = 5$.

(d) Find $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$\lim_{x \to 1} 10^x - 3f'(x) = 10 - 3f'(1) = 10 - 3(2) = 4$

$\lim_{x \to 1} f(x) - \arctan x = 1 - \pi/4$

$\arctan(1) = \pi/4$

$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{4}{1 - \pi/4}$

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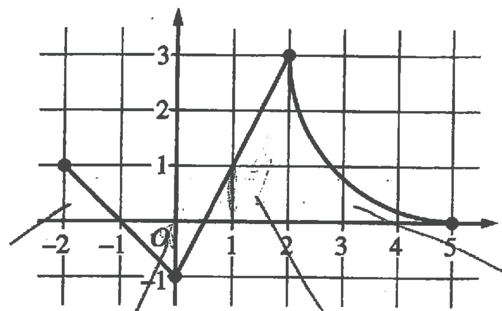
$2+9-\frac{9\pi}{4}$

$\frac{1+3}{2}(1)$

$\frac{1}{2}$



$-\frac{1.5(1)}{2}$ Graph of f $\frac{1.5(3)}{2}$ $9-\frac{\pi 9}{4}$

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx + \int_{-2}^{5} f(x)dx$$

$$\int_{-6}^{5} f(x)dx - \int_{-2}^{5} f(x)dx = \int_{-6}^{-2} f(x)dx$$

$$7 - \left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right) = \boxed{-2 + \frac{9\pi}{4}}$$

$\frac{8}{4} + 9 - \frac{9\pi}{4}$

$11 - \frac{9\pi}{4}$

$7 - 11 + \frac{9\pi}{4} =$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$$2\int_{3}^{5} f'(x)dx + \int_{3}^{5} 4\,dx$$

$$2\left[f(x)\right]_3^5 + \left[4x\right]_3^5$$

$$2(f(5) - f(3)) + 20 - 12$$

$$2(0 - 3 + \sqrt{5}) + 8$$

$2(-3+\sqrt{5}) + 8$

$-6 + 2\sqrt{5} + 8$

$\boxed{2 + 2\sqrt{5}}$

Continue question 3 on page 15.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$+ \rightarrow -$

There are critical numbers at $x = -1, 1.5$ and 5.

$g'(x) = \frac{d}{dx} \int_{-2}^{x} f(t)dt$

$g'(x) = f(x) = 0$ at $x = -1, 1.5, 5$

$+ \quad - \quad + \quad -$
$-1 \quad 1.5 \quad 5$

$g(-1) = \int_{-2}^{-1} f(t)dt = \frac{1}{2}$

$g(5) = \int_{-2}^{5} f(t)dt = 11 - \frac{9\pi}{4}$

Since $g(5) > g(-1)$, the absolute maximum value of g is $11 - \frac{9\pi}{4}$ at $t = 5$. This value is an end point and it ~~may~~ change from positive to negative, alluding to a maximum (absolute) value. It is at a critical value

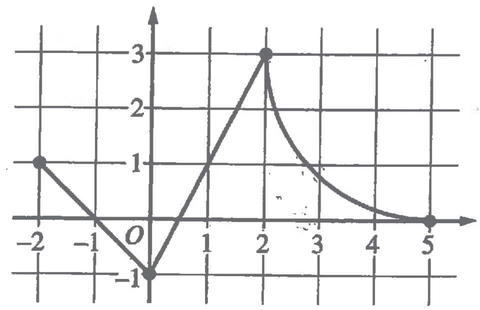
(d) Find $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$f'(1) = \frac{3+1}{2-0} = 2$

plug in 1

$\frac{10^1 - 3f'(1)}{f(1) - \tan^{-1}(1)} = \frac{10 - 3(2)}{1 - \frac{\pi}{4}} = \frac{4}{\frac{\pi}{4}} = \boxed{\frac{1}{\pi}}$

3C 1of2



Graph of f

$x^2+y^2=r^2$
$x^2+y^2=3^2$
$3^2+(3-\sqrt{5})^2=9$

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$\int_{-2}^{5} f(x)\,dx = \frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) - \frac{1}{2}(1)(\frac{1}{2}) + \frac{1}{2}(\frac{3}{2})(3) +$

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

$\int_3^5 2f'(x) + 4 \; dx$

$[f(x)]^2 + 4x \Big|_3^5$

$[f(5)]^2 + 4(5) - ([f(3)]^2 + 4(3))$

$0 + 20 - (3-\sqrt{5})^2 - 12$

$8 - 14 + 6\sqrt{5} \rightarrow \boxed{-6 + 6\sqrt{5}}$

$(3-\sqrt{5})(3-\sqrt{5})$

$9 - 3\sqrt{5} - 3\sqrt{5} + 5$

$14 - 6\sqrt{5}$

Continue question 3 on page 15.

NO CALCULATOR ALLOWED

3C 2of2

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

$g'(x) = \frac{d}{dx}\int_{-2}^{x} f(t)\,dt$

$g'(x) = f(x) = 0$

$x = -1, 0.5, 5$

$+ \quad \cap \quad - \quad \cup \quad + \quad x \quad +$ $g'(x)$

$-1 \quad 0.5 \quad 5$

$g(-1) = \int_{-2}^{-1} f(t)\,dt$

(d) Find $\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$. $\rightarrow \frac{10^1 - 3f'(1)}{f(1) - \arctan(1)} = \frac{10 - 3(2)}{1 - \frac{\pi}{4}} = \frac{4}{\frac{4-\pi}{4}}$

$\frac{4}{1} \cdot \frac{4}{4-\pi} = \boxed{\frac{16}{4-\pi}}$

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem it is given that the function f is continuous on the interval $[-6, 5]$. The portion of the graph of f corresponding to $-2 \le x \le 5$ consists of two line segments and a quarter of a circle, as shown in an accompanying figure. It is noted that the point $(3, 3 - \sqrt{5})$ is on the quarter circle.

In part (a) students were asked to evaluate $\int_{-6}^{-2} f(x)\,dx$, given that $\int_{-6}^{5} f(x)\,dx = 7$. A response should demonstrate the integral property that $\int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx = \int_{-6}^{5} f(x)\,dx$ and use the interpretation of the integral in terms of the area between the graph of f and the x-axis to evaluate $\int_{-2}^{5} f(x)\,dx$ from the given graph.

In part (b) students were asked to evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$. A response should demonstrate the sum and constant multiple properties of definite integrals, together with an application of the Fundamental Theorem of Calculus that gives $\int_{3}^{5} f'(x)\,dx = f(5) - f(3)$.

In part (c) students were asked to find the absolute maximum value for the function g given by $g(x) = \int_{-2}^{x} f(t)\,dt$ on the interval $-2 \le x \le 5$. A response should demonstrate calculus techniques for optimizing a function, starting by applying the Fundamental Theorem of Calculus to obtain $g'(x) = f(x)$, and then using the supplied portion of the graph of f to find critical points for g and to evaluate g at these critical points and the endpoints of the interval.

In part (d) students were asked to evaluate $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$. A response should demonstrate the application of properties of limits, using the supplied portion of the graph of f to evaluate $\lim_{x \to 1} f(x)$ and $\lim_{x \to 1} f'(x)$.

For part (a) see LO FUN-6.A/EK FUN-6.A.2, LO FUN-6.A/EK FUN-6.A.1. For part (b) see LO FUN-6.B/EK FUN-6.B.2. For part (c) see LO FUN-5.A/EK FUN-5.A.2, LO FUN-4.A/EK FUN-4.A.3. For part (d) see LO LIM-1.D/EK LIM-1.D.2. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 3A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the first point was earned with the statement of the property of definite integrals $\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$ in line 1. The second point was earned with $\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$ given for $\int_{-2}^{5} f(x)\,dx$ in line 2. The third point was earned with the answer

Question 3 (continued)

$7 - \left[\frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right]$ in line 2. Numerical simplification of the expression is not required. In part (b) the response earned the first point with $2 \cdot [[f(5) + 2(5)] - [f(3) + 2(3)]]$ in line 4. Note that $2 \cdot [f(x) + 2x]_3^5$ in line 3 is not sufficient to earn the first point. The second point was earned with the answer of $2 \cdot [(0 + 10) - ((3 - \sqrt{5}) + 6)]$ in the last line. Numerical simplification is not required. In part (c) the first point was earned with $g'(x) = f(x)$ in line 1 on the left. The second point was earned in line 3 on the left with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values are required for the third point. The response earned the third point by declaring the absolute maximum value of $11 - \frac{9\pi}{4}$ in line 9 and justifying this answer with the labeled values of g for both critical points and both endpoints. The statement at the top right that references the EVT (Extreme Value Theorem) is not required for the point. In part (d) the point was earned with the answer $\dfrac{4}{1 - \frac{\pi}{4}}$ in the last line. Numerical simplification is not required, and the work presented in lines 1 and 2 is not required to earn the point.

Sample: 3B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no point in part (d). In part (a) the first point was earned with the statement of the property of definite integrals $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$ in line 1. The second point was earned with $\left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right)$ given for $\int_{-2}^{5} f(x)\,dx$ in line 3. The third point would have been earned with the answer $7 - \left(\frac{1}{2} - \frac{3}{4} + \frac{9}{4} + \left(9 - \frac{9\pi}{4}\right)\right)$ in line 3. The numerical simplification to $-2 + \frac{9\pi}{4}$ in line 3 is incorrect, so the third point was not earned. In part (b) the response earned the first point with $2(f(5) - f(3)) + 20 - 12$ in line 3 on the left. Note that $2[f(x)]_3^5 + [4x]_3^5$ in line 2 on the left is not sufficient to earn the first point. The second point would have been earned for $2(0 - 3 + \sqrt{5}) + 8$ in line 4 on the left. Numerical simplification is not required. The boxed answer $2 + 2\sqrt{5}$ is correct, so the second point was earned. In part (c) the first point was earned with $g'(x) = f(x)$ in line 2 on the left. The inclusion of "$= 0$" is not required to earn the point. The second point was earned in line 2 on the left with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values presented are not considered for the second point. The absolute maximum value of $11 - \frac{9\pi}{4}$ is identified in line 4 on the left and declared as the absolute maximum in lines 5 and 6. The third point was not earned because of an insufficient justification. An incorrect critical point is declared at $x = 1.5,$ and the sign chart presented without explanation is not a sufficient justification for elimination of the second critical point. In part (d) the point would have been earned with the answer $\dfrac{10 - 3(2)}{1 - \frac{\pi}{4}}.$ Numerical simplification is not required, though the result of $\frac{1}{\pi}$ is incorrect. The point was not earned.

Question 3 (continued)

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the property of definite integrals that is required is not stated, so the first point was not earned. Although the response begins to calculate $\int_{-2}^{5} f(x)\,dx$, the work is incomplete, and the second point was not earned. The response is not eligible for the third point. In part (b) the antiderivative of $2f'(x)$ is reported incorrectly as $[f(x)]^2$ in line 2 on the left. The Fundamental Theorem of Calculus is not applied correctly, so the first point was not earned. Because the use of the Fundamental Theorem of Calculus is incorrect, the response is not eligible for the second point. In part (c) the first point was earned in line 2 with $g'(x) = f(x)$. The inclusion of "$= 0$" in line 2 is not required to earn the point. The second point was earned in line 3 with $x = -1$ identified as a candidate. The second point only requires this single candidate; the other values presented are not considered for the second point. An absolute maximum value of g is not given, so the third point was not earned. Note that the sign chart without explanation is not a sufficient justification. In part (d) the point would have been earned with the answer $\frac{10 - 3(2)}{1 - \frac{\pi}{4}}$ in line 1. Numerical simplification is not required, though the boxed result of $\frac{16}{4 - \pi}$ is correct and earned the point.

2019

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AP® Calculus AB

Sample Student Responses and Scoring Commentary

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Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$2 : \begin{cases} 1 : \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3 : \begin{cases} 1 : \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1 : \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $$\frac{dh}{\sqrt{h}} = -\frac{1}{10}\, dt$$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\, dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

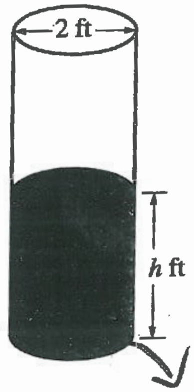
$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

NO CALCULATOR ALLOWED

4A 1of2



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$h=4$

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{4} = \frac{-1}{5}$

$\frac{dV}{dt} = ?$

$r=1$

$\frac{dr}{dt} = 0$

$\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h(2r)\left(\frac{dr}{dt}\right)\right]$

$\frac{dV}{dt} = \pi\left[(1)\left(\frac{-1}{5}\right) + (4)(2)(0)\right]$

$\frac{dV}{dt} = \frac{-\pi}{5}\ ft^3/sec$

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$h=3$

$\frac{d^2h}{dt^2}=?$

$\frac{-1}{10}h^{1/2}$

$\frac{d^2h}{dt^2} = \frac{-1}{20}h^{-1/2}\frac{dh}{dt} = \frac{-1}{20\sqrt{3}}\left(\frac{-1}{10}\sqrt{3}\right) = \frac{1}{200}$

The rate of change of height Is increasing since $\frac{d^2h}{dt^2}$ at $h=3$ Is positive.

$\frac{1}{200}$

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\int h^{-1/2}dh = \int \frac{-1}{10}dt$

$2h^{1/2} = \frac{-1}{10}t + C$

$h^{1/2} = \frac{-1}{20}t + C$

$h = \left(\frac{-1}{20}t + C\right)^2$

$5 = \left(\frac{-1}{20}(0) + c\right)^2$

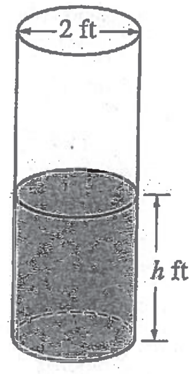
$C = \sqrt{5}$

$h = \left(\frac{-1}{20}t + \sqrt{5}\right)^2$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

4B
1 of 2



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

for $h = 4$:

$$\frac{dV}{dt} = \pi(1)^2 \cdot \left(-\frac{1}{10}\sqrt{4}\right) = \boxed{-\frac{\pi}{5} \text{ feet}^3/\text{s}}$$

Continue question 4 on page 17.

4B 2 of 2

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

at $h=3$, $\frac{dh}{dt} = -\frac{1}{10}\sqrt{3}$, which is negative, so the amount of water is decreasing

---

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{1}{\sqrt{h}}dh = -\frac{1}{10}dt$

$\int \frac{1}{\sqrt{h}}dh = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\frac{t}{10} + C$

$2\sqrt{5} = -\frac{0}{10} + c$

$c = 2\sqrt{5}$

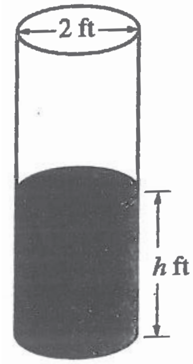
$2\sqrt{h} = -\frac{t}{10} + 2\sqrt{5}$

$\sqrt{h} = -\frac{t}{20} + \sqrt{5}$

$\boxed{h = \left(-\frac{t}{20} + \sqrt{5}\right)^2}$

NO CALCULATOR ALLOWED

4C 1 of 2



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$\frac{dv}{dt} = ?$ $h=4$

$V=\pi r^2 h$

$r=1$

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{h}$

$V=\pi r^2 h$

$\frac{dV}{dt} = 2\pi r \frac{dh}{dt}$

$\frac{dv}{dt} = 2\pi(1)\left(-\frac{1}{10}\sqrt{4}\right)$

$\frac{dv}{dt} = 2\pi \cdot -\frac{2}{10} = -\frac{2\pi}{5}$ ft^3 persecond

Do not write beyond this border.

Continue question 4 on page 17.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$\frac{dh}{dt}$

$r(t) = \frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$r'(t) = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}}$

$r'(3) = -\frac{1}{10} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{3}} = -\frac{1}{20\sqrt{3}} < 0$

When the height of the water is 3 feet the rate of change of height of the water is decreasing because $r'(t) < 0$.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$\frac{1}{\sqrt{h}} dh = -\frac{1}{10} dt$

$2h^{1/2} + c_1 = -\frac{1}{10} t + c_2$

$h = \sqrt{-\frac{1}{20} t + c}$

$5 = \sqrt{-\frac{1}{20}(0) + c}$

$c = 25$

$h = \sqrt{-\frac{1}{20} t + 25}$

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context for this problem is a cylindrical barrel with a diameter of 2 feet that contains collected rainwater, some of which drains out through a valve in the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h},$ where h is measured in feet, and t is measured in seconds.

In part (a) students were asked to find the rate of change of the volume of water in the barrel with respect to time when $h = 4$ feet. A response should use the geometric relationship between the volume V of water in the barrel and height h and incorporate the given expression for $\frac{dh}{dt}.$

In part (b) students were asked to determine whether the rate of change of the height of water in the barrel is increasing or decreasing when $h = 3$ feet. A response should demonstrate facility with the chain rule to differentiate $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ with respect to time to obtain $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}.$ Because $\frac{d^2h}{dt^2} > 0,$ a response should conclude that the rate of change of the height of the water in the barrel is increasing.

In part (c) students were given that the height of the water is 5 feet at time $t = 0$ and then asked to use the technique of separation of variables to find an expression for h in terms of $t.$ A response should demonstrate the application of separation of variables to solve the differential equation $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$ for h and then incorporate the initial condition that $h(0) = 5$ to find the particular solution $h(t)$ to the differential equation.

For part (a) see LO CHA-3.D/EK CHA-3.D.1, LO CHA-3.E/EK CHA-3.E.1. For part (b) see LO FUN-4.E/EK FUN-4.E.2. For part (c) see LO FUN-7.D/EK FUN-7.D.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-7.E/EK FUN-7.E.1. This problem incorporates all four Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 2: Connecting Representations, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 4A
Score: 9

The response earned 9 points: 2 points in part (a), 3 points in part (b), and 4 points in part (c). In part (a) the response earned the first point with the presentation of a correct expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h(2r)\left(\frac{dr}{dt}\right)\right],$ in line 1 on the right. The second point would have been earned with $\frac{dV}{dt} = \pi\left[(1)\left(\frac{-1}{5}\right) + (4)(2)(0)\right]$ in line 2 on the right. Although numerical simplification is not required, the response simplifies the expression in line 3 on the right and adds units to produce $-\frac{\pi}{5}$ $\text{ft}^3/\text{sec}.$ Thus the second point was earned. In part (b) the response presents a correct second derivative of h with respect to $t,$ $\frac{d^2h}{dt^2} = -\frac{1}{20}h^{-\frac{1}{2}}\frac{dh}{dt},$ in line 1 on the right and earned both the first and second points. The response earned the

Question 4 (continued)

third point in lines 2, 3, and 4 on the right with "[t]he rate of change of height is increasing since $\frac{d^2h}{dt^2}$ at $h = 3$ is positive." In part (c) the response earned the first point with a correct separation of variables $\int h^{-\frac{1}{2}}\, dh = \int \frac{-1}{10}\, dt$ in line 1. The correct antiderivatives, $2h^{\frac{1}{2}}$ and $-\frac{1}{10}t,$ are presented in line 2, and the response earned the second point. In lines 2 and 5, the response includes a constant of integration and uses the initial condition $h(0) = 5$ by substituting 0 for t and 5 for $h.$ The response earned the third point. The response solves for h in terms of t and earned the fourth point with $h = \left(\frac{-1}{20}t + \sqrt{5}\right)^2$ in line 7.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), and 4 points in part (c). In part (a) the response earned the first point with the presentation of a correct expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt},$ while handling r as a constant. The second point would have been earned with $\frac{dV}{dt} = \pi(1)^2\left(-\frac{1}{10}\sqrt{4}\right)$ in line 3. Although numerical simplification is not required, the response simplifies the expression in line 3 and adds units to produce $-\frac{\pi}{5}$ feet^3/s. Thus the second point was earned. In part (b) the response does not include the derivative of $\frac{dh}{dt},$ so the first and second points were not earned. Because there is no second derivative, the response is not eligible for the third point. In part (c) the response earned the first point with a correct separation of variables $\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$ in line 1 on the left. The correct antiderivatives, $2\sqrt{h}$ and $-\frac{t}{10},$ are presented in line 3 on the left, and the response earned the second point. In lines 3 and 4 on the left, the response includes a constant of integration and uses the initial condition $h(0) = 5$ by substituting 0 for t and 5 for $h.$ The response earned the third point. The response solves for h in terms of t and earned the fourth point with $h = \left(-\frac{t}{20} + \sqrt{5}\right)^2$ in the box in line 3 on the right.

Sample: 4C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the response presents an incorrect expression for the derivative of V with respect to $t,$ $\frac{dV}{dt} = 2\pi r \frac{dh}{dt},$ in line 3 on the left. The first point was not earned, and this error makes the response not eligible for the second point. In part (b) the response defines $r(t) = \frac{dh}{dt}$ in line 1. The response earned the first point with $r'(t) = -\frac{1}{10} \cdot \frac{1}{2} h^{-\frac{1}{2}}$ in line 2. The expression is identified as the second derivative of h with respect to $t;$ however, $r'(t)$ does not

Question 4 (continued)

include a factor of $\frac{dh}{dt}$. Thus the second point was not earned, and the response is not eligible for the third point.

In part (c) the response earned the first point with a correct separation of variables $\frac{1}{\sqrt{h}}\, dh = -\frac{1}{10}\, dt$ in line 2 on the left. The correct antiderivatives, $2h^{\frac{1}{2}}$ and $-\frac{1}{10}t,$ are presented in line 3 on the left, and the response earned the second point. In line 3 on the left, the response includes constants of integration. The response incorrectly solves for h in terms of t in line 4 before using the initial condition $h(0) = 5$ in line 5. The resulting expression $h = \sqrt{-\frac{1}{20}t + C}$ in line 4 on the left is incorrect. Thus the response is not eligible for the third and fourth points.

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Sample Student Responses and Scoring Commentary

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Free Response Question 5

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Question 5

(a) $\int_0^2 (h(x) - g(x))\, dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right) dx$

$$= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$$

$$= \left(\left(12 - \frac{2}{3}\right) - (-4 + 0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0 + 0)\right)$$

$$= 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$$

The area of R is $\frac{44}{3}$.

4 :
- 1 : integrand
- 1 : antiderivative of $3\cos\left(\frac{\pi}{2}x\right)$
- 1 : antiderivative of remaining terms
- 1 : answer

(b) $\int_0^2 A(x)\, dx = \int_0^2 \frac{1}{x+3}\, dx$

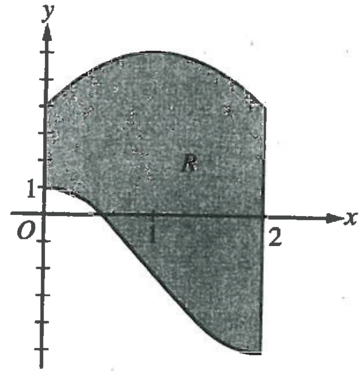
$$= \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$$

The volume of the solid is $\ln 5 - \ln 3$.

2 :
- 1 : integral
- 1 : answer

(c) $\pi \int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2 \right) dx$

3 :
- 1 : limits and constant
- 1 : form of integrand
- 1 : integrand



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

$$\text{Area} = \int_0^2 h(x) - g(x)\,dx$$

$$= \int_0^2 6 - 2(x-1)^2\,dx - \int_0^2 -2 + 3\cos\left(\tfrac{\pi}{2}x\right)dx$$

$$= 6x - \tfrac{2}{3}(x-1)^3 \Big|_0^2 - \left(-2x + \tfrac{6}{\pi}\sin\left(\tfrac{\pi}{2}x\right)\right)\Big|_0^2$$

$$= \left(\left[12 - \tfrac{2}{3}(1)\right] - \left[0 + \tfrac{2}{3}\right]\right) - \left(\left[-4 + \tfrac{6}{\pi}(0)\right] - \left[0 + 0\right]\right)$$

$$= 12 - \tfrac{4}{3} + 4 = 16 - \tfrac{4}{3}$$

Continue question 5 on page 19.

5A 2 of 2

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$V = \int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx = \ln|x+3|\Big|_0^2 = \ln|5| - \ln|3|$$

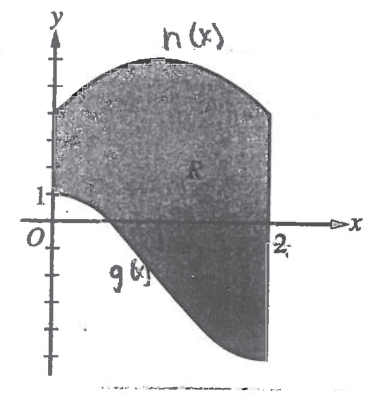
$$= \ln 5 - \ln 3$$

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$V = \int_0^2 \pi\left[(6-g(x))^2 - (6-h(x))^2\right] dx$$

NO CALCULATOR ALLOWED

5B
1 of 2



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

$$\int_0^2 (h(x) - g(x))\,dx$$

$$\int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\tfrac{\pi}{2}x\right)\right)\right)dx$$

$$6x - \tfrac{2}{3}(x-1)^3 \Big]_0^2 - \left(-2x + \tfrac{3\pi}{2}\sin\left(\tfrac{\pi}{2}x\right)\right)\Big]_0^2$$

$$12 - (-4) = 12 + 4 = 16$$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

5B

2 of 2

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$V = \pi \int_0^2 (A(x))dx = \pi \int_0^2 \frac{1}{x+3} dx$$

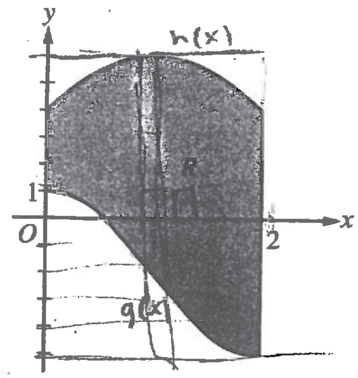
$$V = \pi \cdot \ln|x+3| \Big]_0^2 = \pi (\ln|5| - \ln|3|)$$

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$\pi \int_0^2 (6 - g(x))^2 - (6 - h(x))^2 dx$$

GO ON TO THE NEXT PAGE.

5C 1 of 2



5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

$$A = \int_0^2 (h(x) - g(x))^2 \, dx$$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

5C 2of2

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

$$V = \int_0^2 \frac{1}{x+3} (h(x) - g(x))^2 \, dx$$

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

$$V = \pi \int_0^2 (g(x) - 6)^2 - (h(x) - 6)^2 \, dx$$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem R is identified as the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$.

In part (a) students were asked to find the area of R. A response should demonstrate the area interpretation of definite integrals and compute the area of R as $\int_0^2 (h(x) - g(x))\,dx$. Students should find an antiderivative for $h(x) - g(x)$ and apply the Fundamental Theorem of Calculus to evaluate the integral.

In part (b) students were asked to find the volume of a solid having R as its base and for which at each x, the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. A response should demonstrate that the volume is found by integrating the cross-sectional area function across the interval $0 \le x \le 2$. As before, the Fundamental Theorem of Calculus should be employed to evaluate $\int_0^2 A(x)\,dx$.

In part (c) students were asked to write an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$. A response should demonstrate that the volume is found by integrating the cross-sectional area function across the interval $0 \le x \le 2$. In this case, however, the cross section at x is a "washer" with outer radius $6 - g(x)$ and inner radius $6 - h(x)$, so the area of the cross section at x can be expressed using the familiar formula for the area of a circle.

For part (a) see LO CHA-5.A/EK CHA-5.A.1, LO FUN-6.C/EK FUN-6.C.2, LO FUN-6.B/EK FUN-6.B.3. For part (b) see LO CHA-5.B/EK CHA-5.B.3, LO FUN-6.B/EK FUN-6.B.3. For part (c) see LO CHA-5.C/EK CHA-5.C.4. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes and Practice 4: Communication and Notation.

Sample: 5A
Score: 9

The response earned 9 points: 4 points in part (a), 2 points in part (b), and 3 points in part (c). In part (a) the response earned the first point in line 1 with $\int_0^2 h(x) - g(x)\,dx$. The missing parentheses in the integrand do not impact earning the point. The antiderivative of the cosine term, $-\left(+\frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)$, in line 3 is correct, and the response earned the second point. The antiderivative of the remaining terms, $6x - \frac{2}{3}(x-1)^3 - (-2x)$, in line 3 is correct, and the response earned the third point. The response earned the fourth point in the last line with $16 - \frac{4}{3}$. Note that the response could have ended at line 4 because numerical simplification is not required for the fourth point. Although this response does so, evaluation of trigonometric functions is also not required for the fourth

Question 5 (continued)

point. In part (b) the response earned the first point with $\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$ in line 1. The first definite integral is sufficient to earn the point. The second point was earned with the answer $\ln 5 - \ln 3$ in line 2. Note that the absolute value is not needed in this case because on the interval $[0, 2]$, $x + 3 > 0$. In part (c) the response earned all 3 points with the expression $\int_0^2 \pi\left[(6 - g(x))^2 - (6 - h(x))^2\right] dx$ because the limits, constant, and integrand are correct.

Sample: 5B
Score: 6

The response earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the response earned the first point in line 1 with $\int_0^2 (h(x) - g(x))\,dx$. The antiderivative of the cosine term, $-\left(+\frac{3\pi}{2}\sin\left(\frac{\pi}{2}x\right)\right)$, in line 3 is incorrect because the factor of $\frac{3\pi}{2}$ should be $\frac{6}{\pi}$. The second point was not earned. The antiderivative of the remaining terms, $6x - \frac{2}{3}(x-1)^3 - (-2x)$, in line 3 is correct, and the response earned the third point. Because both the first point and 1 of the second and third points were earned, the response is eligible for the fourth point if the answer is consistent with previous work. The response has an error on the left side of the equation in line 4 because the expression $12 - (-4)$ should be $\left(12 - \frac{4}{3}\right) - (-4)$. The fourth point was not earned. In part (b) the response earned the first point with $\int_0^2 (A(x))\,dx$ in line 1. The multiplication by π is incorrect and produces an incorrect answer. The second point was not earned. In part (c) the response earned all 3 points with the expression $\pi\int_0^2 (6 - g(x))^2 - (6 - h(x))^2\,dx$ because the limits, constant, and integrand are correct. The missing parentheses in the integrand do not impact earning the point.

Sample: 5C
Score: 3

The response earned 3 points: no points in part (a), no points in part (b), and 3 points in part (c). In part (a) the response has a definite integral $\int_0^2 (h(x) - g(x))^2\,dx$ with an incorrect integrand, so the first point was not earned. A response that includes a definite integral that when evaluated does not represent the area of R is not eligible for points in part (a). As a result, although this response presents no additional work, the response is not eligible for any points in part (a). In part (b) the response has a definite integral $\int_0^2 \frac{1}{x+3}(h(x) - g(x))^2\,dx$ with an incorrect factor of $(h(x) - g(x))^2$ in the integrand. The first point was not earned, and the response is not eligible for the second point because of the form of the integrand presented. In part (c) the response earned all 3 points with the expression $\pi\int_0^2 (g(x) - 6)^2 - (h(x) - 6)^2\,dx$ because the limits, constant, and integrand are correct. Because each of the two terms in the integrand are squared, this response presents an integrand that is equivalent to $(6 - g(x))^2 - (6 - h(x))^2$. The missing parentheses in the integrand do not impact earning the point.

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Sample Student Responses and Scoring Commentary

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Question 6

(a) $h'(2) = \frac{2}{3}$	1 : answer
(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$ $a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$	$3 : \begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$
(c) Because h is differentiable, h is continuous, so $\lim_{x \to 2} h(x) = h(2) = 4$. Also, $\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$, so $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$. Because $\lim_{x \to 2} \left(x^2 - 4\right) = 0$, we must also have $\lim_{x \to 2} \left(1 - (f(x))^3\right) = 0$. Thus $\lim_{x \to 2} f(x) = 1$. Because f is differentiable, f is continuous, so $f(2) = \lim_{x \to 2} f(x) = 1$. Also, because f is twice differentiable, f' is continuous, so $\lim_{x \to 2} f'(x) = f'(2)$ exists. Using L'Hospital's Rule, $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$ Thus $f'(2) = -\frac{1}{3}$.	$4 : \begin{cases} 1 : \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$
(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x \to 2} g(x) = g(2) = 4$ and $\lim_{x \to 2} h(x) = h(2) = 4$. Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x \to 2} k(x) = 4$. Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$. Thus k is continuous at $x = 2$.	1 : continuous with justification

6A 1of2

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$y = \frac{2}{3} \cdot (x-2) + 4$

point $(2,4)$ slope $\frac{2}{3}$

$h'(2) = 2/3$

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$a'(x) = h'(x) \cdot 3x^3 + 9x^2 h(x)$

$a'(2) = h'(2) \cdot 3(2)^3 + 9(2)^2 h(2)$

$a'(2) = \frac{2}{3} \cdot 3 \cdot 2^3 + 9 \cdot 2^2 \cdot 4$

Continue question 6 on page 21.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$\lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \dfrac{2x}{-3(f(x))^2 \cdot f'(x)}$

$h(2)=4$

$\therefore \lim_{x\to 2} h(x)$ MUST equal 4

$\lim_{x\to 2} x^2-4 = 0$

$\lim_{x\to 2} 1-(f(x))^3 = 0$ if $f(2)=1$

$\therefore$ Indeterminate form $\frac{0}{0}$ and L'Hopital's rule applies SUCH THAT $f(2)=1$

$= \dfrac{2(2)}{-3(1)^2 \cdot f'(2)}$

$= \dfrac{4}{-3 \cdot f'(2)}$

$f'(2)$ MUST equal $-1/3$ so that $\lim_{x\to 2} h(x) = 4$

$= \dfrac{4}{-3 \cdot -1/3}$

$= \dfrac{4}{1}$

$\lim_{x\to 2} h(x) = 4$ such that $f(2)=1$ and $f'(2) = -1/3$

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

g and h given as twice-differentiable $\therefore$ g and h are continuous for $1<x<3$

$g(x) \leq k(x) \leq h(x)$

$\lim_{x\to 2} g(x) = 4$ because g continuous and $g(2)=4$

$\lim_{x\to 2} h(x) = 4$ because h continuous and $h(2)=4$

By Squeeze theorem, $\lim_{x\to 2} k(x) = 4$

k is between or equal to g and h for $1<x<3$, given $g(2)=4=h(2)$ meaning $k(2)$ must equal 4, and $\lim_{x\to 2} k(x)=4$ so $k(x)$ is continuous at $x=2$

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$$y = 4 + \frac{2}{3}(x-2)$$
$$y - 4 = \frac{2}{3}(x-2)$$

$$h'(2) = \frac{2}{3}$$

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$$a'(x) = h(x)(9x^2) + (3x^3)h'(x)$$
$$a'(2) = h(2)(9(2)^2) + (3(2)^3)h'(2)$$
$$a'(2) = 4(9(2)^2) + (3(2)^3)\left(\frac{2}{3}\right)$$

Continue question 6 on page 21.

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(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$$\lim_{x \to 2} \frac{x^2-4}{1-(f(x))^3} = 4$$

$$\lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = 4$$

$$\frac{2(2)}{-3(f(2))^2 f'(2)} = 4$$

$$\frac{4}{-3(f(2))^2 f'(2)} = 4$$

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

K is continuous at x=2 given that k(x) is less than or equal too h(x) on 1< x < 3 and greater than or equal too g(x) on 1 < x < 3, so k(x) must simply be equal to g(x) and h(x) at x=2 since both g(x) and h(x) = 4 at x=2

6C 1 of 2

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$h'(2) = 2/3$

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

$a'(x) = (3x^3)(h'(x)) + (9x^2)(h(x))$

$a'(2) = (3(2)^3)(h'(2)) + (9(2)^2)(h(2))$

$= 8 \cdot h'(2) + 36 \cdot 4$

$= 8 \cdot h'(2) + 144$

$= 8 \cdot \frac{2}{3} + 144$

$\frac{16}{3} + \frac{432}{3}$

$\frac{448}{3}$

Continue question 6 on page 21.

NO CALCULATOR ALLOWED

6C 2 of 2

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

$$\frac{2x}{-3(f(x))^2 f'(x)}$$

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

k is continuos at 2 because using IVT, it states that there is a value for $1<x<3$ that equals that due to MVT. $g(2)=h(2)=4$ so yes.

GO ON TO THE NEXT PAGE.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem introduces three twice-differentiable functions f, g, and h. It is given that $g(2) = h(2) = 4$, and the line $y = 4 + \frac{2}{3}(x - 2)$ is tangent at $x = 2$ to both the graph of g and the graph of h.
In part (a) students were asked to find $h'(2)$. A response should demonstrate the interpretation of the derivative as the slope of a tangent line and answer with the slope of the line $y = 4 + \frac{2}{3}(x - 2)$.

In part (b) the function a given by $a(x) = 3x^3 h(x)$ is defined, and students were asked for an expression for $a'(x)$ and the value of $a'(2)$. A response should demonstrate facility with the product rule for differentiation.

In part (c) it is given that the function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$ and that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Students were then asked to find $f(2)$ and $f'(2)$. A response should observe that the differentiability of h implies that h is continuous so that $\lim_{x \to 2} h(x) = h(2) = 4$. Because $\lim_{x \to 2} \left(x^2 - 4\right) = 0$, and $\lim_{x \to 2} h(x)$ can be evaluated, as is given, it must be that $\lim_{x \to 2} \left(1 - (f(x))^3\right) = 0$, as well. Using properties of limits, students could conclude that $\lim_{x \to 2} f(x) = 1$. Finally, an application of L'Hospital's Rule to $\lim_{x \to 2} \dfrac{x^2 - 4}{1 - (f(x))^3}$, combined with the chain rule to differentiate $(f(x))^3$, yields an equation that can be solved for $f'(2)$.

In part (d) students were given that $g(x) \le h(x)$ for $1 < x < 3$ and that k is a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Students were asked to decide, with justification, whether k is continuous at $x = 2$. A response should observe that the differentiability of g and h implies that these functions are continuous, so the limits as x approaches 2 of each of g and h match the value $g(2) = h(2) = 4$. From the inequality $g(2) \le k(2) \le h(2)$ it follows that $k(2) = 4$, and the squeeze theorem applies to show that k is continuous at $x = 2$.

For part (a) see LO CHA-2.C/EK CHA-2.C.1. For part (b) see LO FUN-3.B/EK FUN-3.B.1. For part (c) see LO LIM-2.A/EK LIM-2.A.2, LO LIM-4.A/EK LIM-4.A.2. For part (d) see LO LIM-1.E/EK LIM.1.E.2. This problem incorporates the following Mathematical Practices: Practice 1: Implementing Mathematical Processes, Practice 3: Justification, and Practice 4: Communication and Notation.

Sample: 6A
Score: 9

The response earned 9 points: 1 point in part (a), 3 points in part (b), 4 points in part (c), and 1 point in part (d). In part (a) the response earned the point for the value of $h'(2) = \frac{2}{3}$ in line 3. In part (b) the response earned both the first and second points for the derivative $a'(x) = h'(x) \cdot 3x^3 + 9x^2 h(x)$ in line 1. The response earned the third

Question 6 (continued)

point for the numerical expression $a'(2) = \frac{2}{3} \cdot 3 \cdot 2^3 + 9 \cdot 2^2 \cdot 4$ in line 3. Numerical simplification is not required. In part (c) the response earned the first point for the verbal connection "$\lim_{x \to 2} h(x)$ MUST equal 4" in lines 2 and 3 on the right. The response would have earned the second point for the equation $\lim_{x \to 2} 1 - (f(x))^3 = 0$ in line 3 on the left and presenting $f(2) = 1$ in line 3 on the left. The response correctly restates $f(2) = 1$ in line 6 on the left and earned the second point with the final restatement of $f(2) = 1$ in the circled statement in the last line. The response earned the third point for L'Hospital's Rule with the expression $\lim_{x \to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)}$ in line 1 in the middle. The response would have earned the fourth point with "$f'(2)$ MUST equal $\frac{-1}{3}$" in line 4 on the right. The response earned the fourth point with the restatement of $f'(2) = \frac{-1}{3}$ in the circled statement in the last line. In part (d) the response earned the point for concluding that $k(x)$ is continuous at $x = 2$ in line 8 with the following justification: stating g and h are continuous in line 1, evaluating $\lim_{x \to 2} g(x) = 4$ in line 4 and $\lim_{x \to 2} h(x) = 4$ in line 5, concluding $\lim_{x \to 2} k(x) = 4$ in line 6, and concluding $k(2) = 4$ because $g(2) = 4 = h(2)$ in lines 7 and 8. Although not required, the response correctly states use of the squeeze theorem.

Sample: 6B
Score: 6

The response earned 6 points: 1 point in part (a), 3 points in part (b), 2 points in part (c), and no point in part (d). In part (a) the response earned the point for the value of $h'(2) = \frac{2}{3}$ in line 1 on the right. In part (b) the response earned the first and second points for the derivative $a'(x) = h(x)\left(9x^2\right) + \left(3x^3\right)h'(x)$ in line 1. The response earned the third point for the numerical expression $a'(2) = 4\left(9(2)^2\right) + \left(3(2)^3\right)\left(\frac{2}{3}\right)$ in line 3. Numerical simplification is not required. In part (c) the response earned the first point for the equation $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$ in line 1 on the left. Although not required for the first point, application of L'Hospital's Rule resulting in the equation $\lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = 4$ in line 1 on the right would also have earned the first point. The response did not earn the second point because no value is given for $f(2)$. The response earned the third point for the expression $\lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)}$ in line 1 on the right. The response did not earn the fourth point because no value is given for $f'(2)$. In part (d) the response correctly concludes that k is continuous at $x = 2$. The response did not earn the point because the justification is not sufficient. The response does not state that g and h are continuous, does not evaluate $\lim_{x \to 2} g(x) = 4$ and $\lim_{x \to 2} h(x) = 4$, and does not conclude $\lim_{x \to 2} k(x) = 4 = k(2)$.

Question 6 (continued)

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), no points in part (c), and no point in part (d). In part (a) the response earned the point for the value of $h'(2) = \frac{2}{3}$. In part (b) the response earned the first and second points for the derivative $a'(x) = \left(3x^3\right)(h'(x)) + \left(9x^2\right)(h(x))$ in line 1. The response did not earn the third point because of an incorrect simplification in the numerical expression of $a'(2)$ with rewriting $\left(3(2)^3\right)$ in line 2 as 8 in line 3. In part (c) the response did not earn the first point because there is no connection between $\lim_{x\to 2} h(x)$ and $4.$ The response did not earn the second point because no value is given for $f(2).$ Although the response attempts to apply L'Hospital's Rule in line 1, the response did not earn the third point because of a lack of limit notation: The use of $\lim_{x\to 2}$ is not presented with the quotient. The response did not earn the fourth point because no value is given for $f'(2).$ In part (d) the response correctly concludes that k is continuous at $2.$ The response did not earn the point because the justification is not sufficient. The response attempts to apply both the "IVT" (Intermediate Value Theorem) and the "MVT" (Mean Value Theorem) rather than the squeeze theorem.